Ciências ULisboa

Diffusion of vorticity & Boundary layers

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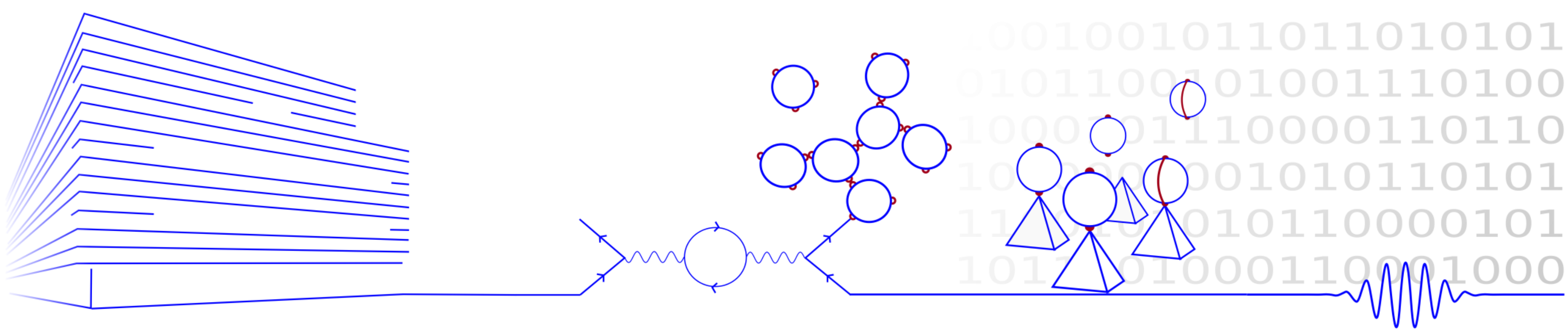
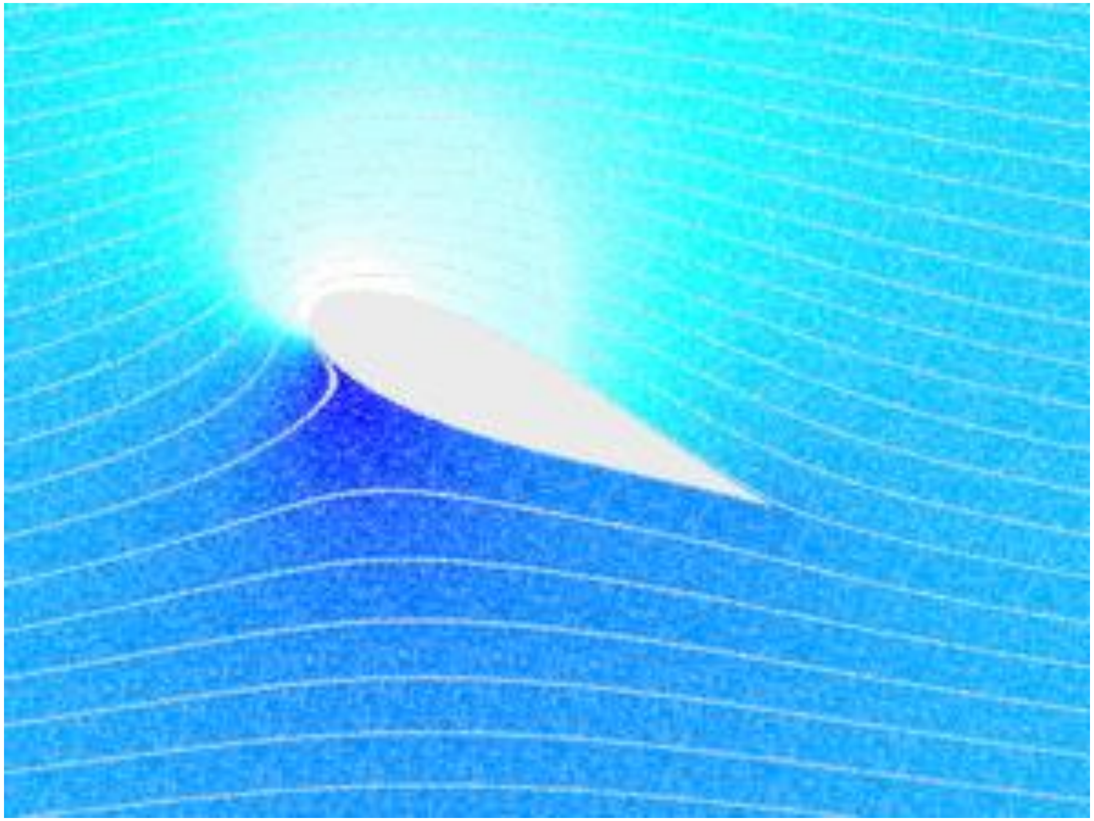


Overview

- There are at least two flow situations in which the viscous term in the Navier–Stokes equation can be neglected.
- The first occurs in high Reynolds number regions of flow where net viscous forces are known to be negligible compared to inertial and/or pressure forces; we call these inviscid regions of flow.
- The second situation occurs when the vorticity is negligibly small; we call these irrotational or potential regions of flow.
- In either case, removal of the viscous terms from the Navier–Stokes equation yields the Euler equation.
- There are some serious deficiencies associated with application of the Euler equation to practical flow problems. High on the list of deficiencies is the inability to specify the no-slip condition at solid walls.

Çengel 10.6



Irrotational flow

- Irrotational flow around a wing.
- The solution may be obtained from potential flow theory.
- This particular solution is obtained as the sum of three elementary solutions: free stream, line source and line vortex.
- Lift force proportional to the circulation and free stream velocity.
- Drag force zero.

- By the mid-1800s, the Navier–Stokes equation was known, but couldn't be solved except for flows of very simple geometries.
- Meanwhile, mathematicians were able to obtain beautiful analytical solutions of the Euler equation and of the potential flow equations for flows of complex geometry, but their results were often physically meaningless.

Nº do Prandtl (Revisão)

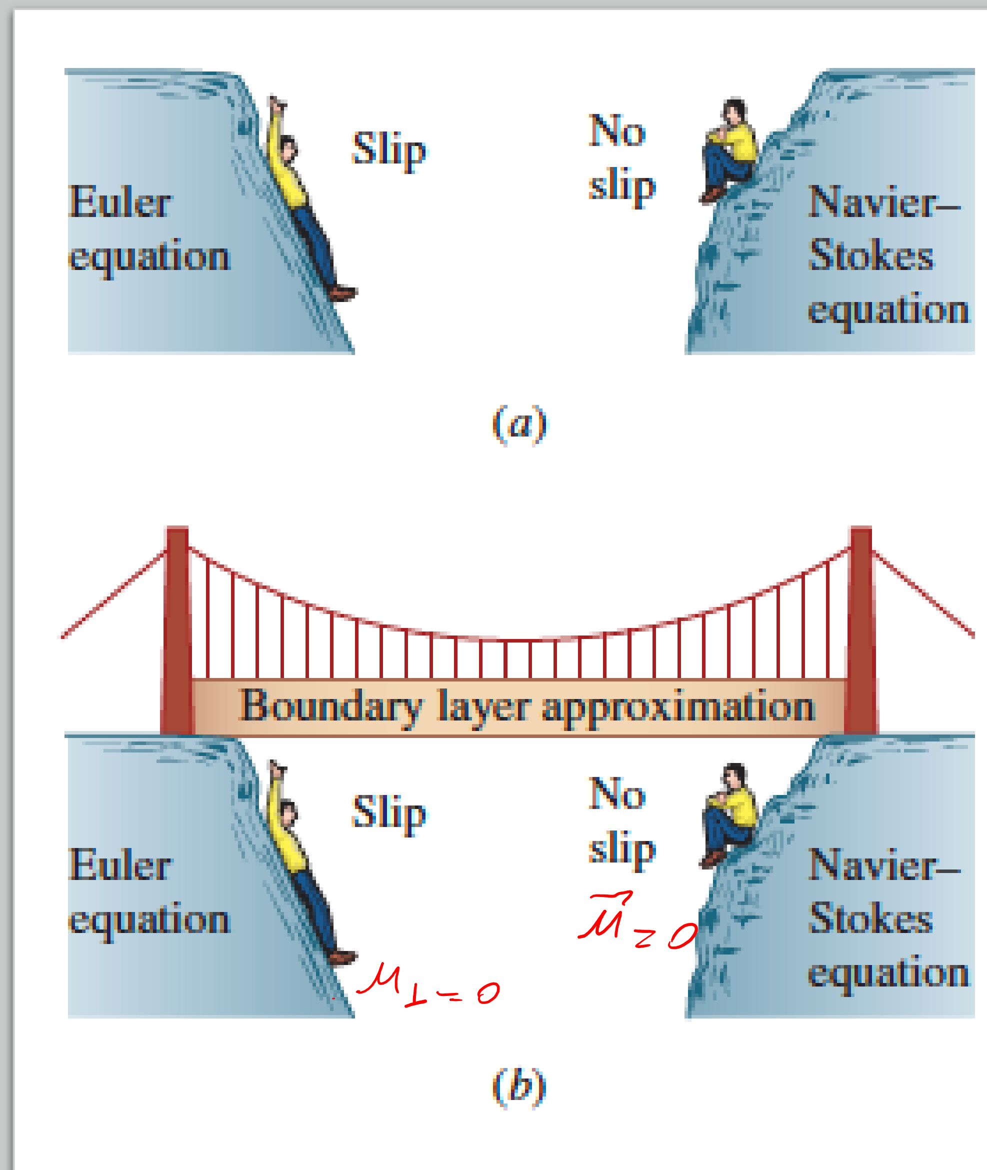
$$Pr = \frac{L \cdot V}{D}$$

D — Coef. de difusão

- A major breakthrough in fluid mechanics occurred in 1904 when Ludwig Prandtl (1875–1953) introduced the boundary layer approximation.
- Prandtl's idea was to divide the flow into two regions: an outer flow region that is inviscid and/or irrotational, and an inner flow region called a boundary layer—a very thin region of flow near a solid wall where viscous forces and rotationality cannot be ignored.
- In the outer flow region, the continuity and Euler equations apply to obtain the outer flow velocity field, and the Bernoulli equation to obtain the pressure field. Alternatively, if the outer flow region is irrotational, we may use potential flow techniques.

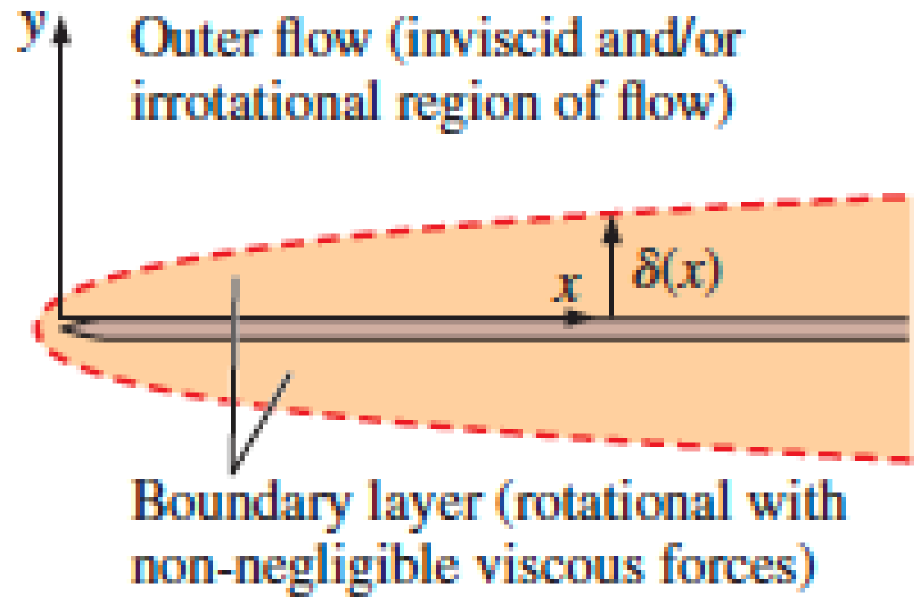
The boundary layer

Prandtl introduced the boundary layer approximation to bridge the gap between the Euler equation and the Navier–Stokes equation, and between the slip condition and the no-slip condition at solid walls.

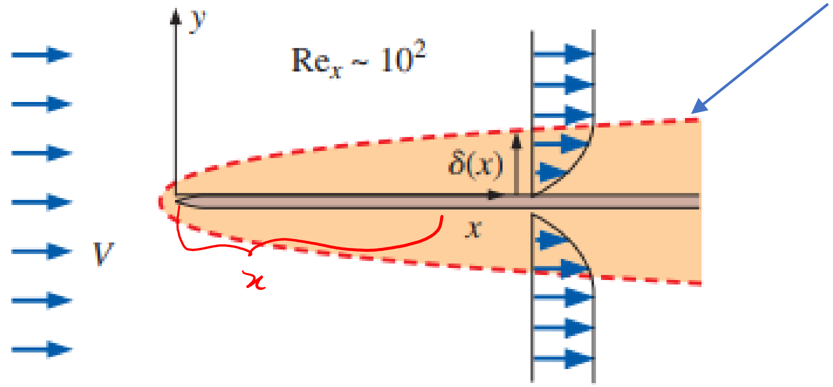


The idea

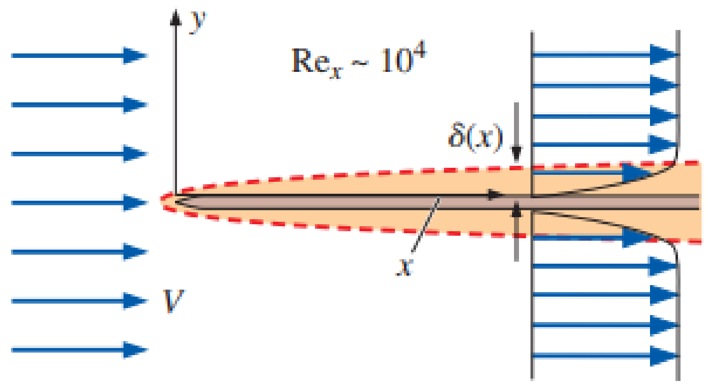
We solve for the outer flow region first, and then fit in a thin boundary layer in regions where vorticity and viscous forces cannot be neglected.



The larger the Reynolds number, the thinner the boundary layer along the plate at a given *x*-location



(*a*)



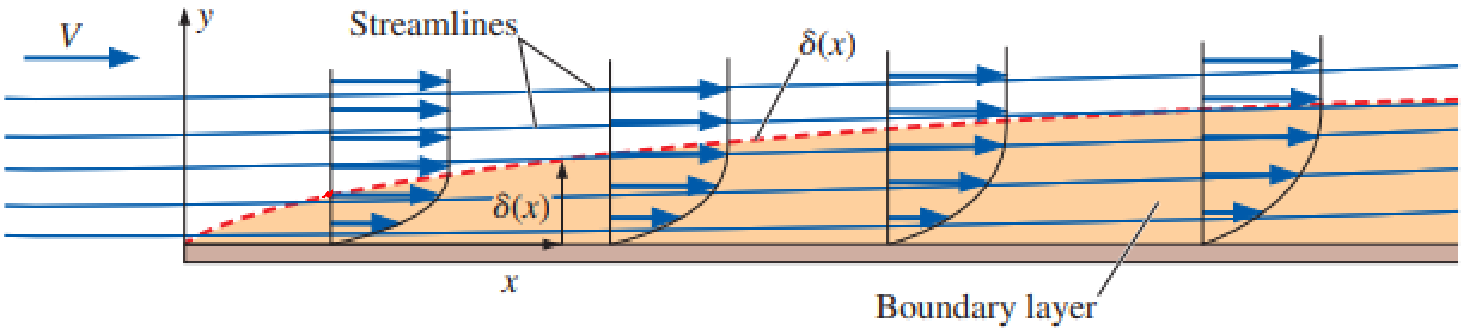
(*b*)

Note: this is not a streamline.

Reynolds number along a flat plate:

$$Re_x = \frac{\rho V x}{\mu} = \frac{V x}{\nu}$$

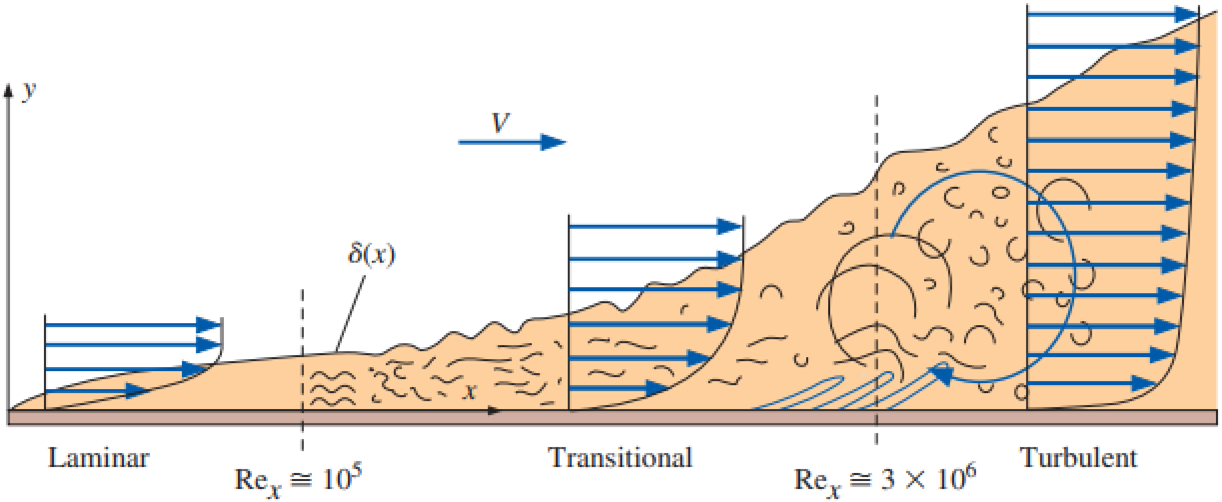
Laminar to turbulent transition



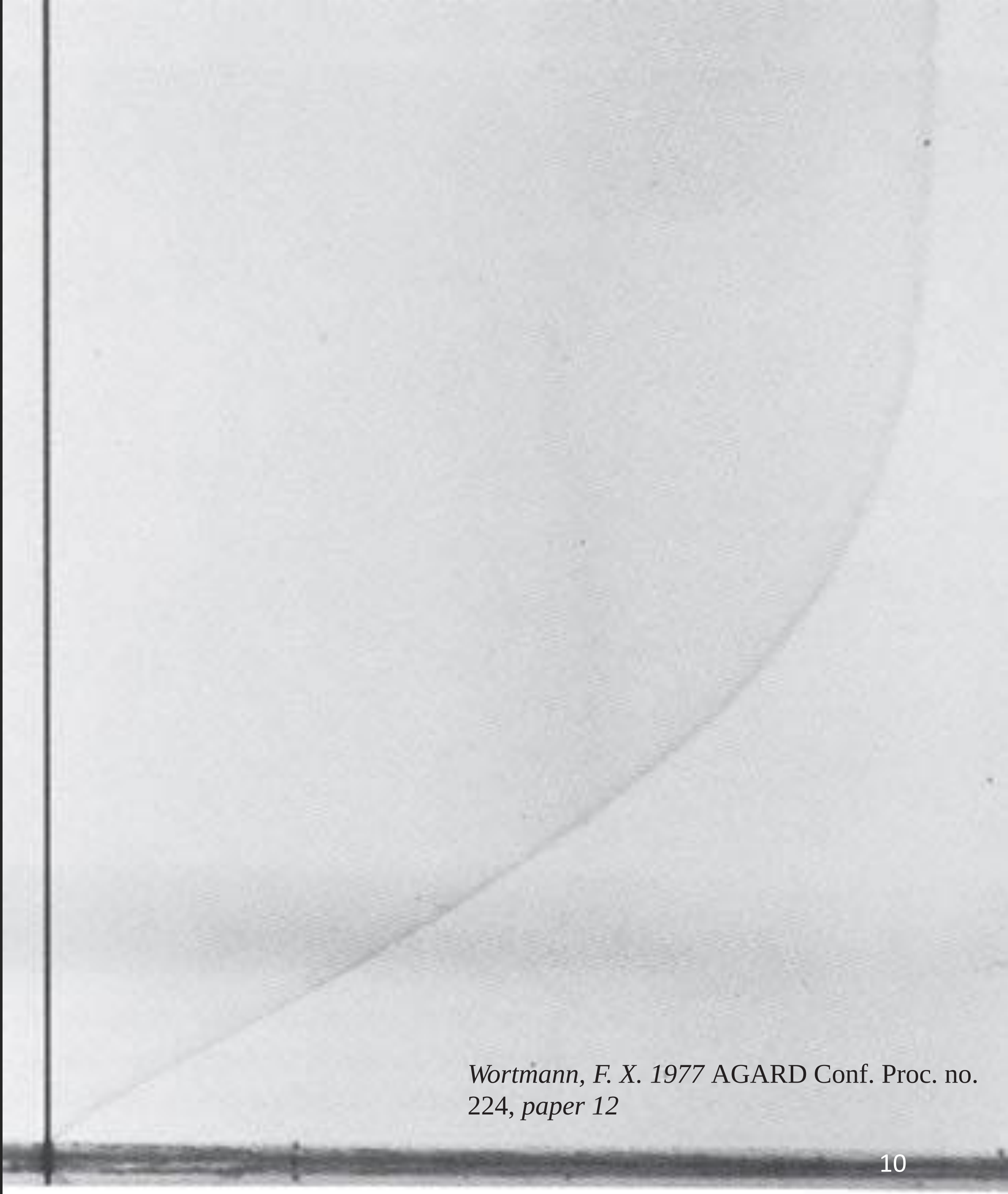
critical Reynolds number, $Re_{x,\text{ critical}} \cong 1 \times 10^5$,

$$Re_{x,\text{ transition}} \cong 3 \times 10^6$$

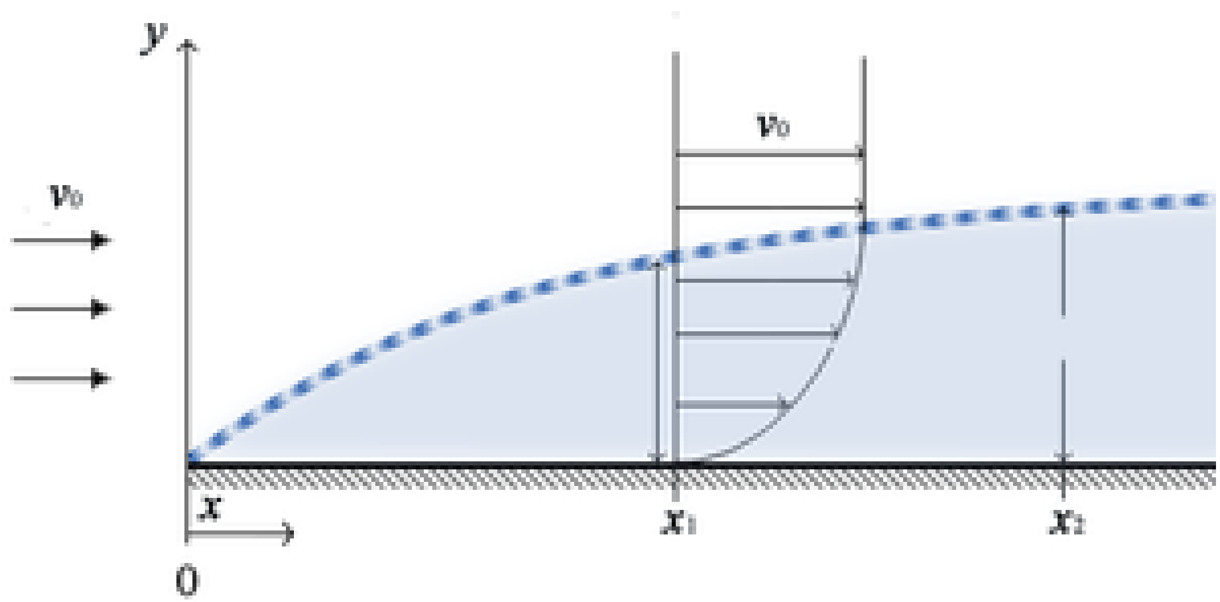
Analogy:



Photograph of a velocity profile of a uniform stream over a flat plate



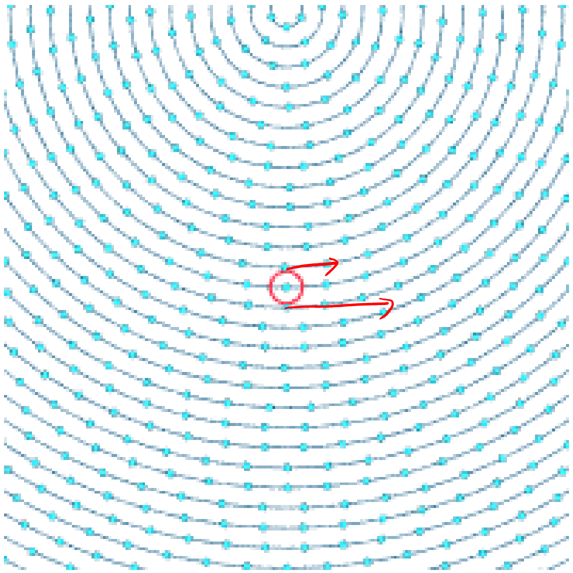
Wortmann, F. X. 1977 AGARD Conf. Proc. no. 224, *paper 12*



Negligible viscosity or irrotational flow cannot be assumed near solid boundaries, such as the case of the airplane wing.

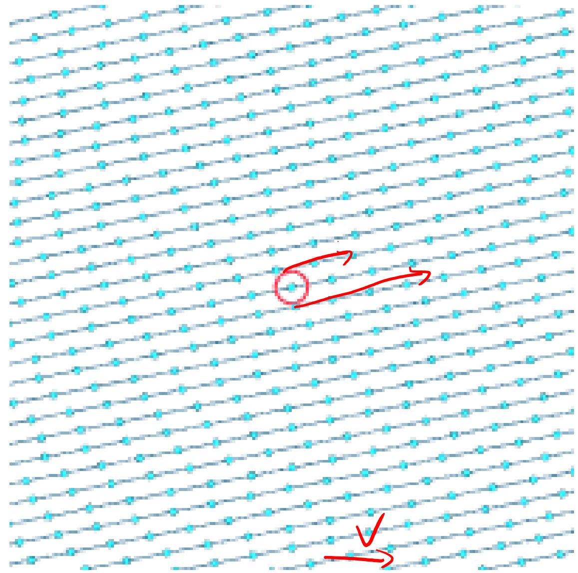
Rigid-body-like vortex

$v \propto r$



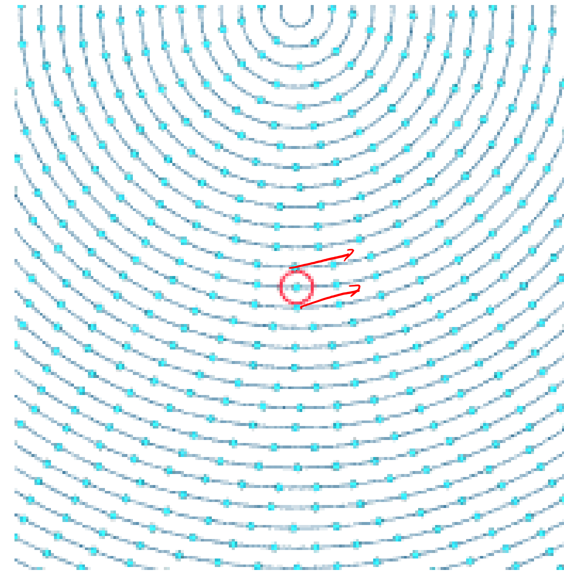
Balde

Parallel flow with shear



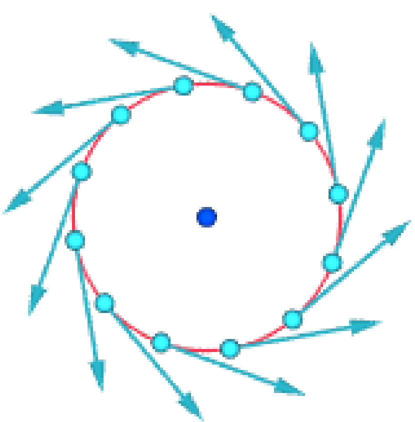
Irrotational vortex

$v \propto 1/r$

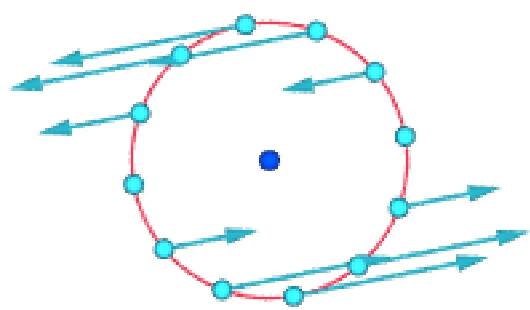


Tornado

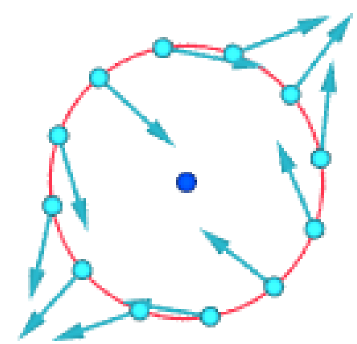
Relative velocities (magnified) around the highlighted point



Vorticity ≠ 0



Vorticity ≠ 0



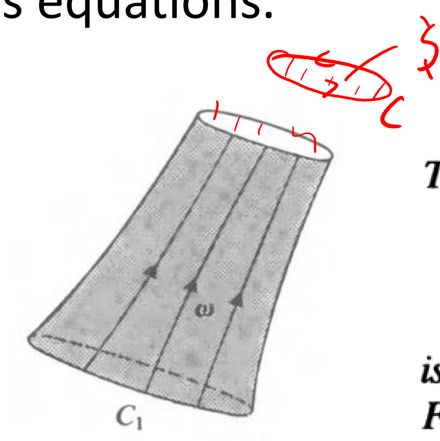
Vorticity = 0

Vorticity and lines of vorticity

$$\nabla \cdot (\underbrace{\nabla \times \vec{V}}_{\omega}) = 0$$

- Since $\vec{\Omega} = \nabla \times \vec{V}$ its divergence is zero, i.e. $\boxed{\nabla \cdot \vec{\Omega} = 0}$.

- The vorticity is a solenoidal field with lines of vorticity (like streamlines) parallel to its direction and density proportional to its magnitude.
- Dynamics of the lines of vorticity differs in the Euler and Navier-Stokes equations.



The quantity

$$\Gamma = \int_S \boldsymbol{\omega} \cdot \boldsymbol{n} \, \mathrm{d}S = \int_c \vec{u} \cdot d\vec{l} \qquad (5.6)$$

is the same for all cross-sections S of a vortex tube. Furthermore, Γ is independent of time.

Euler fluid

Chap. 5, Acheson

Euler equation

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla\left(\frac{p}{\rho} + \chi\right)$$

Identity

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = (\nabla \wedge \boldsymbol{u}) \wedge \boldsymbol{u} + \nabla(\tfrac{1}{2}\boldsymbol{u}^2)$$

$$\nabla \times \left[\frac{\partial \boldsymbol{u}}{\partial t} + \underbrace{(\nabla \wedge \boldsymbol{u})}_{\vec{\omega}} \wedge \boldsymbol{u} = -\nabla\left(\frac{p}{\rho} + \tfrac{1}{2}\boldsymbol{u}^2 + \chi\right) \right]$$

Application of the curl

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \wedge (\boldsymbol{\omega} \wedge \boldsymbol{u}) = 0$$

Identity

$$\nabla \wedge (\boldsymbol{F} \wedge \boldsymbol{G}) = (\boldsymbol{G} \cdot \nabla)\boldsymbol{F} - (\boldsymbol{F} \cdot \nabla)\boldsymbol{G} + \boldsymbol{F}(\nabla \cdot \boldsymbol{G}) - \boldsymbol{G}(\nabla \cdot \boldsymbol{F})$$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} + \boldsymbol{\omega}\underbrace{\nabla \cdot \boldsymbol{u}}_{=0} - \boldsymbol{u}\underbrace{\nabla \cdot \boldsymbol{\omega}}_{\nabla \cdot \nabla \times (\ldots) = 0} = 0$$

We have

$$\underbrace{\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega}}_{\frac{D\vec{\omega}}{Dt}} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}$$

Vorticity equation

$$\boxed{\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}}$$

$$\omega_x \frac{\partial}{\partial x}\hat{x} + \omega_y \frac{\partial}{\partial y}\hat{y} + \omega_z \frac{\partial}{\partial z}\hat{z}$$

If the flow is 2D

$$\boldsymbol{u} = [u(x, y, t), v(x, y, t), 0]$$

$$\boldsymbol{\omega} = (0, 0, \omega)$$

$u_z = cte$

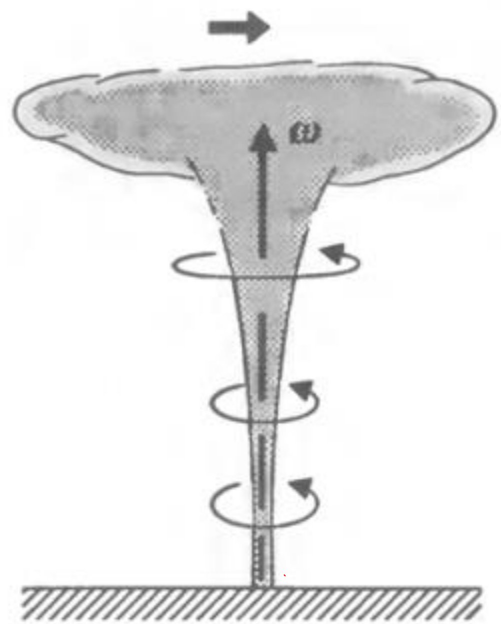
Then

$$(\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} = \omega \frac{\partial \boldsymbol{u}}{\partial z} = 0$$

It follows that

$$\frac{D\omega}{Dt} = 0$$

In the two-dimensional flow of an ideal fluid subject to a conservative body force g the vorticity ω of each individual fluid element is conserved.



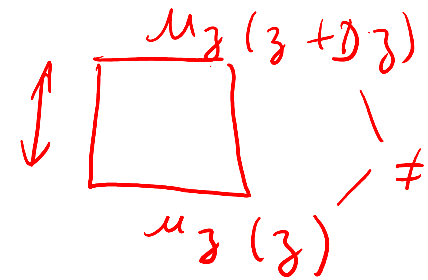
Vortex lines in the z-direction

$$\boldsymbol{\omega} \doteqdot \omega \boldsymbol{k}$$

Z-component:

$$\frac{D\boldsymbol{\omega}}{Dt} \doteqdot \omega \frac{\partial \boldsymbol{u}}{\partial z}$$

$$\frac{D\omega}{Dt} \doteqdot \omega \frac{\partial w}{\partial z}$$



The vorticity of a fluid element increases with time if $\partial w / \partial z > 0$.

If the fluid elements are being stretched in the z-direction, it leads to an intensification of the local vorticity field.

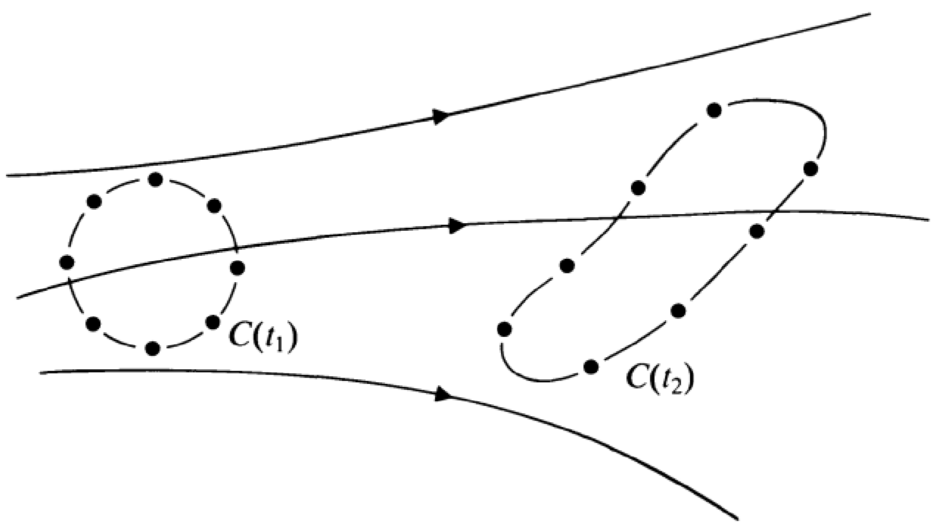
Kelvin circulation theorem

THEOREM. *Let an inviscid, incompressible fluid of constant density be in motion in the presence of a conservative body force* $\mathbf{g} = -\nabla\chi$ *per unit mass. Let* $C(t)$ *denote a closed circuit that consists of the same fluid particles as time proceeds* (*Fig.* 5.1). *Then the circulation*

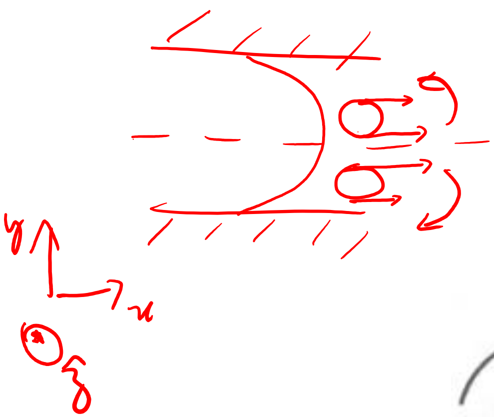
$$\Gamma = \int_{C(t)} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} \tag{5.1}$$

round $C(t)$ *is independent of time.*

$$\frac{D\Gamma}{Dt} = 0$$



The inviscid equations of motion enter the proof only in helping to evaluate a line integral round C, so if viscous forces happened to be important elsewhere in the flow, i.e. off the curve C, this would not affect the conclusion that Γ remains constant round C.



Navier-Stokes: Viscosity drives the diffusion of vorticity

Faber, chap. 10

To find what difference viscosity makes, we need to repeat the above analysis using the Navier–Stokes equation as our starting point, rather than the Euler equation. The viscous term on the left-hand side of (6.25) is $-\eta\boldsymbol{\nabla}\wedge\boldsymbol{\Omega}$, and the curl of this, since $\boldsymbol{\nabla}\cdot\boldsymbol{\Omega}=0$, is $\eta\nabla^2\boldsymbol{\Omega}$. Hence we now have

$$\frac{\mathrm{D}\boldsymbol{\Omega}}{\mathrm{D}t} = (\boldsymbol{\Omega}\cdot\boldsymbol{\nabla})\boldsymbol{u} + \underbrace{\frac{\eta}{\rho}}_{=\nu}\nabla^2\boldsymbol{\Omega}.$$

Apart from the $(\boldsymbol{\Omega}\cdot\boldsymbol{\nabla})\boldsymbol{u}$ term, the effects of which are as described above, this is just a three-dimensional diffusion equation for each of the components of $\boldsymbol{\Omega}$; to be more precise, it becomes a three-dimensional equation in the co-moving frame for which $\mathrm{D}\boldsymbol{\Omega}/\mathrm{D}t$ and $\partial\boldsymbol{\Omega}/\partial t$ are the same. Thus vorticity is not permanently embedded if the fluid has viscosity; where $\nabla^2\boldsymbol{\Omega}$ is non-zero it spreads by diffusion, and its *diffusivity* is the kinematic viscosity, $\nu=\eta/\rho$. Since the process described by the diffusion equation always conserves the thing which is diffusing, whether it be dye or heat or whatever, the fact that vorticity is liable to diffuse does not affect our conclusion that lines of vorticity are conserved.

Diffusion Eq.:

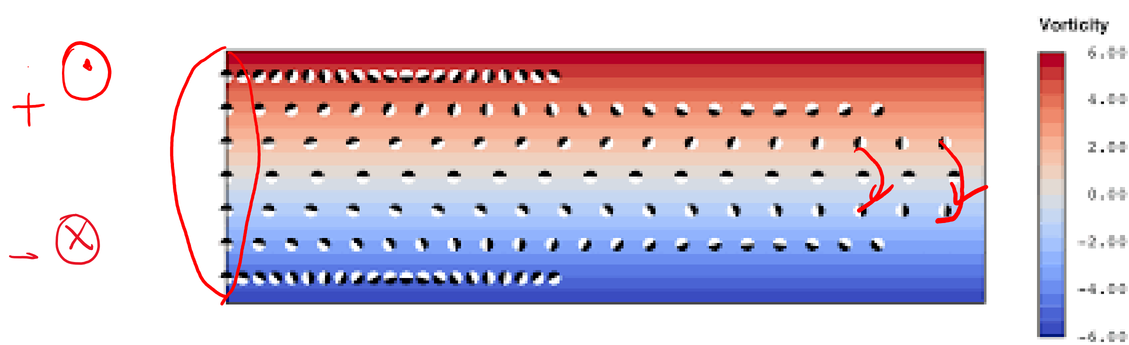
$$\frac{\partial c}{\partial t} = D\nabla^2\phi$$

Example: Poiseuille flow

If, however, the vorticity is positive in region A and negative in an adjoining region B, diffusion from A to B and *vice versa* is bound to result in some degree of cancellation. The lines of vorticity in such situations tend to form closed loops which disappear by collapsing to a point. For example, consider the simple case of a fluid undergoing Poiseuille flow along a straight cylindrical pipe whose axis is the x_3 axis. In the plane $x_2 = 0$, say, Ω_1 and Ω_3 both vanish, while

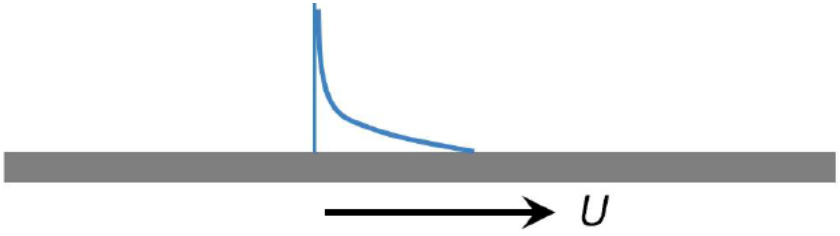
$$\Omega_2 = -\frac{\partial u_3}{\partial x_1} = -\frac{x_1}{2\eta}\nabla_3 p$$

Thus $\boldsymbol{\Omega}$ changes sign on the axis in the plane $x_2 = 0$, and it also does so in the plane $x_1 = 0$ where Ω_1 is the non-vanishing component; evidently the lines of vorticity are closed circular loops coaxial with the pipe. Now the direction in which the lines of vorticity diffuse is determined by the sign of $\partial\Omega/\partial r$. Because this is positive we should picture the loops as diffusing inwards, to smaller values of radius r, and ultimately collapsing on the axis. We should therefore picture the surface of the fluid, where it is in contact with the solid wall of the pipe, as a vorticity source at which new loops are continuously created to replace those which collapse.



Sudden motion of an infinite flat plane (revisited)

Flow above a solid wall at y = 0. Initially, the fuid is at rest. At time t = 0, the boundary starts to move with velocity U in the x direction.



The velocity field is

$$\mathbf{u} = (u(y,t), 0, 0).$$

and the Navier-Stokes equation

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \mu \nabla^2 \mathbf{u},$$

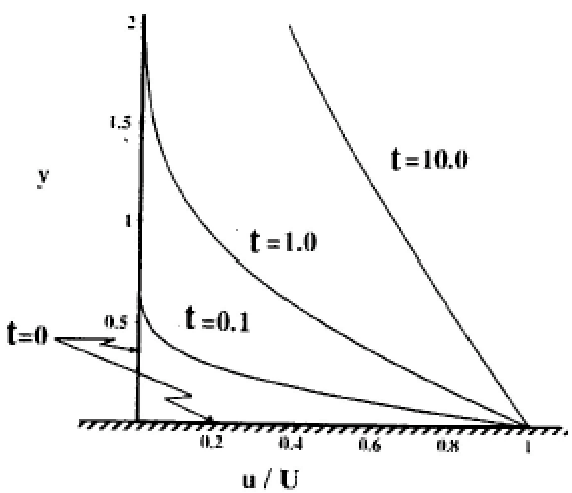
reduces to

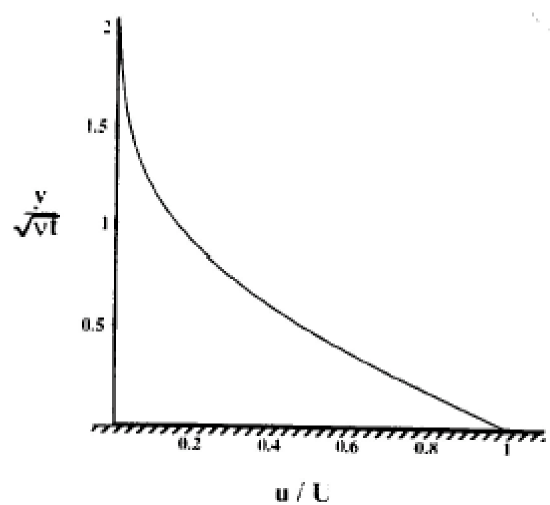
$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2},$$

- Boundary conditions: u = U on y = 0 and U → 0 as y → ∞.
- We also impose the initial condition: u = 0 at t = 0.
- The velocity u(x, t) thus satisfies the 1-D diffusion equation with diffusivity $\upsilon = \frac{\mu}{\varrho}$, where υ is the kinematic viscosity.
- Similarity solution is

$$u(y,t) = U\left[1 - \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)\right].$$

The velocity $u(y,t)$ will be approximately zero wherever $y/2\sqrt{\nu t}$ is large. In addition, for a fixed value of y, the velocity will remain less than $0.01U$ until a time t such that $y \approx 4\sqrt{\nu t}$. Hence, at time t, the fluid is only moving within a narrow region of thickness $4\sqrt{\nu t}$. This narrow region is called the *viscous boundary layer*. Note that the boundary layer thickness is independent of U.





Diffusion of vorticity from the surface to the fluid

Let us now return to the case of the flow above a boundary that is set in motion at time $t = 0$. Initially, the vorticity is zero everywhere, except at $y = 0$ where the fluid velocity jumps from U to 0. At time t, the velocity is given by equation (4.2). The vorticity ω reads:

$$\omega = -\frac{\partial u}{\partial y} = \frac{U}{\sqrt{\pi\nu t}} \exp\left(-\frac{y^2}{4\nu t}\right).$$

This is a Gaussian distribution of standard deviation $\sqrt{2\nu t}$. Hence, as times increases, the vorticity gradually spreads away from the boundary over a distance of order $\sqrt{2\nu t}$.

Since

$$u(y,t) = U\left[1 - \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)\right].$$

$$\frac{d}{dz}\operatorname{erf} z = \frac{2}{\sqrt{\pi}} e^{-z^2}$$

SOLUTION OF THE 1D DIFFUSION EQUATION

We seek a *similarity solution*:

$$u(y,t) = f(\eta), \text{ where } \eta = yt^a,$$

for some constant a. Using the chain rule:

$$\frac{\partial}{\partial y} = t^a \frac{d}{d\eta},$$
$$\frac{\partial}{\partial t} = ayt^{a-1}\frac{d}{d\eta},$$

so that equation (4.1) becomes:

$$ayt^{a-1}\frac{df}{d\eta} = \nu t^{2a}\frac{d^2 f}{d\eta^2},$$

and therefore:

$$\frac{d^2 f}{d\eta^2} - \frac{ayt^{-a-1}}{\nu}\frac{df}{d\eta} = 0.$$

For the similarity solution to exist, this equation must only contain y and t in the combination $\eta = yt^a$ and therefore $-a-1 = a$. We get: $a = -\frac{1}{2}$. Solutions thus exists for the similarity variable $\eta = y/\sqrt{t}$ and satisfy:

$$\frac{d^2 f}{d\eta^2} + \frac{\eta}{2\nu}\frac{df}{d\eta} = 0.$$

Substituting $v = df/d\eta$ we have:

$$\frac{dv}{d\eta} = -\frac{\eta}{2\nu}v,$$

which has general solution:

$$v = \frac{df}{d\eta} = A\exp\left(-\frac{\eta^2}{4\nu}\right).$$

Integrating again, we obtain:

$$f = A\int_0^\eta \exp\left(-\frac{\eta^2}{4\nu}\right)d\eta + B.$$

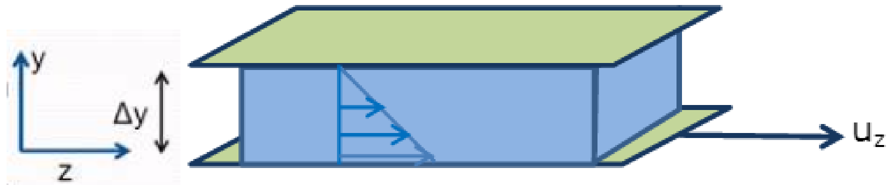
The above integral can be expressed in terms of the error function:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-x^2\right)dx.$$

Substituting $x = \eta/2\sqrt{\nu}$, we have:

$$f = A\sqrt{\nu\pi}\operatorname{erf}\left(\frac{\eta}{2\sqrt{\nu}}\right) + B.$$

Start up of shear flow (parallel plates)



Let us now modify the previous problem by considering the start-up of a shear flow between two parallel plates located at $y = 0$ and $y = h$. Once again, we begin to move the lower plate with velocity U at $t = 0$. The problem is the same as that above except that the boundary condition at infinity is replaced by one at $y = h$. The velocity now satisfies:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}, \tag{4.3}$$

together with the boundary conditions: $u(0,t) = U$ and $u(h,t) = 0$, and the initial condition $u(y,0) = 0$.
First, we observe that the steady solution $u_s = U\,(1 - y/h)$ satisfies the equation at any $t \neq 0$ and the boundary conditions. We then write:

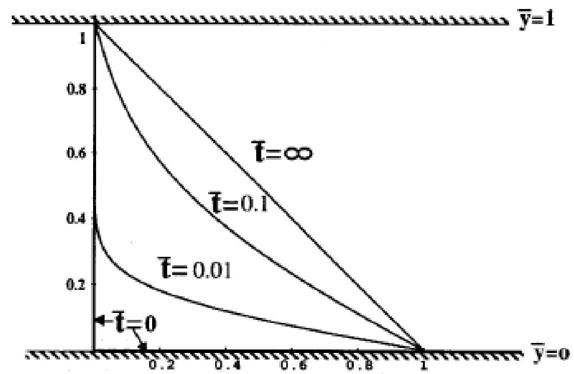
$$u(y,t) = u_s + v(y,t),$$

and seek a separable solution of the form:

$$v(y,t) = T(t)Y(y).$$

Hence, the solution is:

$$u(y,t) = U\left(1-\frac{y}{h}\right) - \frac{2U}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp\left(-\frac{\nu n^2\pi^2}{h^2}t\right)\sin\left(\frac{n\pi y}{h}\right).$$



This flow resembles that of the unbounded plate until the boundary layer grows to the width of the channel. The solution then approaches the steady state u_s. Note that the slowest decaying exponential in the sum corresponds to $n = 1$. As a result, the flow reaches u_s on a time of order $h^2/(\nu\pi^2)$. For water in a 1cm channel, this time is about 10s and scales inversely with ν so that in a fluid of lower viscosity it becomes longer.

SOLUTION OF START UP OF SHEAR FLOW

This gives:

$$YT' = \nu TY'',$$

so that:

$$\frac{Y''}{Y} = \frac{1}{\nu}\frac{T'}{T} = k,$$

where k is the constant of integration. Since u_s takes care of the moving boundary, we want to find solutions satisfying $Y(0) = Y(h) = 0$. We thus choose solutions of the form:

$$Y(y) = \sin\left(\frac{n\pi y}{h}\right),$$

so that:

$$\frac{Y''}{Y} = -\frac{n^2\pi^2}{h^2}.$$

It follows:

$$\frac{T'}{T} = -\frac{\nu n^2\pi^2}{h^2},$$

and so we have separable solutions of the form:

$$v_n = \exp\left(-\frac{\nu n^2\pi^2}{h^2}t\right)\sin\left(\frac{n\pi y}{h}\right).$$

The general solution for u satisfying the boundary conditions is:

$$u(y,t) = U\left(1 - \frac{y}{h}\right) + \sum_{n=1}^{\infty} a_n \exp\left(-\frac{\nu n^2\pi^2}{h^2}t\right)\sin\left(\frac{n\pi y}{h}\right).$$

The initial condition at $t = 0$ requires:

$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi y}{h}\right) = -U\left(1 - \frac{y}{h}\right),$$

for $0 < y < h$. We can determine the a_n using Fourier series properties:

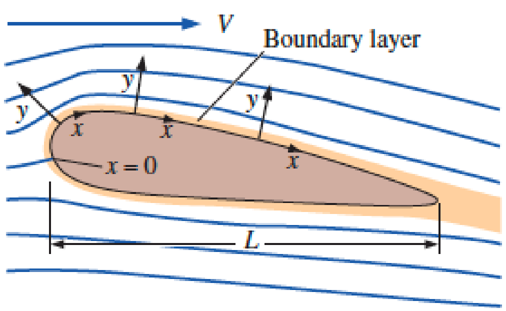
$$a_n = \frac{2U}{h}\int_0^h \left(\frac{y}{h} - 1\right)\sin\left(\frac{n\pi y}{h}\right)dy = -\frac{2U}{n\pi},$$

Hence, the solution is:

$$u(y,t) = U\left(1 - \frac{y}{h}\right) - \frac{2U}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp\left(-\frac{\nu n^2\pi^2}{h^2}t\right)\sin\left(\frac{n\pi y}{h}\right).$$

Boundary layer equations

- We consider steady, two-dimensional flow in the xy-plane in Cartesian coordinates. The methodology can be extended to axisymmetric boundary layers or to three-dimensional boundary layers in any coordinate system.
- We neglect gravity since we are not dealing with free surfaces or with buoyancy-driven flows (free convection flows), where gravitational effects dominate.
- We consider laminar boundary layers; turbulent boundary layer equations are beyond the scope of this course.
- For a boundary layer along a solid wall, we adopt a coordinate system in which x is everywhere parallel to the wall and y is everywhere normal to the wall.
- When we solve the boundary layer equations, we do so at one x-location at a time, using this coordinate system locally, and it is locally orthogonal.

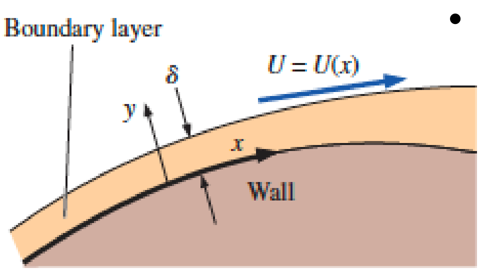


The nondimensionalized Navier–Stokes equation is

$$(\vec{V}^* \cdot \vec{\nabla}^*)\vec{V}^* = -[\mathrm{Eu}]\vec{\nabla}^* P^* + \left[\frac{1}{\mathrm{Re}}\right]\nabla^{*2}\vec{V}^*$$

- The Euler number is of order 1, since pressure differences outside the boundary layer are determined by the Bernoulli equation and ΔP $\sim \rho V^2$.
- V is a characteristic velocity of the outer flow, typically the free-stream velocity for bodies immersed in a uniform flow.
- The characteristic length is L, some characteristic size of the body. For boundary layers, x is of order o L, and Reynolds number is Re_x, usuallly very high.

Redo the nondimensionalization of the equations based on appropriate scales within the boundary layer.



- Since x ~ L, we use L as the scale for distances in the streamwise direction and for derivatives with respect to x. However, this scale is too large for derivatives with respect to y. We use δ for distances in the direction normal to the streamwise direction and for derivatives with respect to y.

- Similarly, we use U as the characteristic velocity, where U is the magnitude of the velocity component parallel to the wall at a location just above the boundary layer. U is in general a function of x.

- Thus, within the boundary layer at some value of x, the orders of magnitude are

$$u \sim U \qquad P - P_\infty \sim \rho U^2 \qquad \frac{\partial}{\partial x} \sim \frac{1}{L} \qquad \frac{\partial}{\partial y} \sim \frac{1}{\delta}$$

- The order of magnitude of velocity component v is obtained from the continuity equation

$\nabla . \vec{u} = 0$

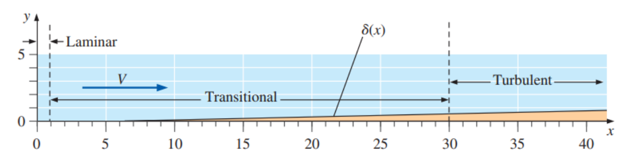
$$\underbrace{\frac{\partial u}{\partial x}}_{\sim U/L} + \underbrace{\frac{\partial v}{\partial y}}_{\sim v/\delta} = 0 \quad \rightarrow \quad \frac{U}{L} \sim \frac{v}{\delta}$$

- Since the two terms have to balance each other, they must be of the same order of magnitude. Thus we obtain the order of magnitude of velocity component v,

$$\boxed{v \sim \frac{U\delta}{L}}$$

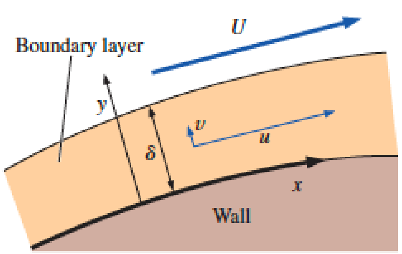
$\delta << L$

$v << U$



On scale

- Since δ/L << 1 in a boundary layer, we conclude that v << u, and the adimensional variables are



$$\rightarrow \quad x^* = \frac{x}{L} \qquad y^* = \frac{y}{\delta} \qquad u^* = \frac{u}{U} \qquad v^* = \frac{vL}{U\delta} \qquad P^* = \frac{P - P_\infty}{\rho U^2}$$

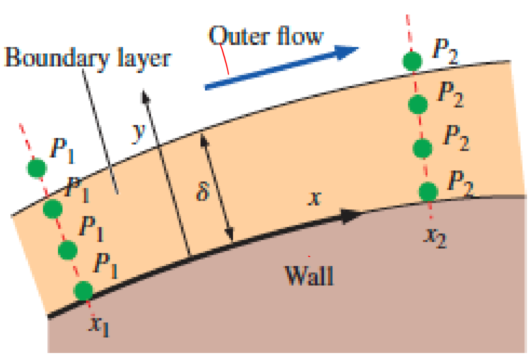
We now consider the x- and y-components of the Navier–Stokes equation. We substitute these nondimensional variables into the y-momentum equation, giving

$$\underbrace{u}_{u^*U}\underbrace{\frac{\partial v}{\partial x}}_{\frac{\partial}{\partial x^*}\frac{v^*U\delta}{L^2}} + \underbrace{v}_{v^*\frac{U\delta}{L}}\underbrace{\frac{\partial v}{\partial y}}_{\frac{\partial}{\partial y^*}\frac{v^*U\delta}{L\delta}} = \underbrace{-\frac{1}{\rho}\frac{\partial P}{\partial y}}_{\frac{1}{\rho}\frac{\partial}{\partial y^*}\frac{P^*\rho U^2}{\delta}} + \underbrace{\nu\frac{\partial^2 v}{\partial x^2}}_{\nu\frac{\partial^2}{\partial x^{*2}}\frac{v^*U\delta}{L^3}} + \underbrace{\nu\frac{\partial^2 v}{\partial y^2}}_{\nu\frac{\partial^2}{\partial y^{*2}}\frac{v^*U\delta}{L\delta^2}}$$

After some algebra

$$\rightarrow \underbrace{u^*\frac{\partial v^*}{\partial x^*}}_{\sim 1} + \underbrace{v^*\frac{\partial v^*}{\partial y^*}}_{\sim 1} = -\overbrace{\left(\frac{L}{\delta}\right)^2}^{\gg 1}\frac{\partial P^*}{\partial y^*} + \overbrace{\left(\frac{\nu}{UL}\right)}^{=Re^{-1} \ll 1}\frac{\partial^2 v^*}{\partial x^{*2}} + \underbrace{\left(\frac{\nu}{UL}\right)}_{Re^{-1} \ll 1}\overbrace{\left(\frac{L}{\delta}\right)^2}^{\gg 1}\frac{\partial^2 v^*}{\partial y^{*2}}$$

The middle term on the r.h.s. is clearly smaller than any other term since $Re_L = UL/\upsilon \gg 1$. For the same reason, the last term on the right is much smaller than the first term on the right. Neglecting these two terms leaves the two terms on the left and the first term on the right. However, since $L \gg \delta$, the pressure gradient is orders of magnitude greater than the advective terms on the left of the equation. Thus, the only term left is the pressure term. Since no other term in the equation can balance that term, we have no choice but to set it to zero. Thus, the nondimensional y-momentum equation is



$$\Rightarrow \frac{\partial P^*}{\partial y^*} \cong 0$$

The pressure across a boundary layer (y-direction) is nearly constant.

Since P is not a function of y, we replace ∂P/∂x by dP/dx, where P is the pressure calculated from the outer flow approximation (using either continuity plus Euler, or the potential flow equations plus Bernoulli). The x-component of the Navier–Stokes equation becomes

$$\underbrace{u}_{u^*U}\underbrace{\frac{\partial u}{\partial x}}_{\frac{\partial}{\partial x^*}\frac{u^*U}{L}} + \underbrace{v}_{v^*\frac{U\delta}{L}}\underbrace{\frac{\partial u}{\partial y}}_{\frac{\partial}{\partial y^*}\frac{u^*U}{\delta}} = \underbrace{-\frac{1}{\rho}\frac{dP}{dx}}_{\frac{1}{\rho}\frac{\partial}{\partial x^*}\frac{P^*\rho U^2}{L}} + \underbrace{\nu\frac{\partial^2 u}{\partial x^2}}_{\nu\frac{\partial^2}{\partial x^{*2}}\frac{u^*U}{L^2}} + \underbrace{\nu\frac{\partial^2 u}{\partial y^2}}_{\nu\frac{\partial^2}{\partial y^{*2}}\frac{u^*U}{\delta^2}}$$

or

$$\underbrace{u^*\frac{\partial u^*}{\partial x^*}}_{\sim 1} + \underbrace{v^*\frac{\partial u^*}{\partial y^*}}_{\sim 1} = \underbrace{-\frac{dP^*}{dx^*}}_{\sim 1} + \underbrace{\left(\frac{\nu}{UL}\right)}_{=Re^{-1} \ll 1}\overset{=0}{\frac{\partial^2 u^*}{\partial x^{*2}}} + \underbrace{\left(\frac{\nu}{UL}\right)}_{Re^{-1} \ll 1}\underbrace{\left(\frac{L}{\delta}\right)^2}_{\gg 1}\frac{\partial^2 u^*}{\partial y^{*2}}$$

The middle term on the right side is orders of magnitude smaller than the terms on the left. What about the last term on the right? If we neglect this term, we throw out all the viscous terms and are back to the Euler equation. Clearly this term must remain. Furthermore, since all the remaining terms are of order unity, the combination of parameters in parentheses in the last term on the right side must also be of order 1,

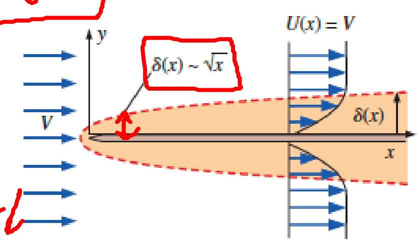
$x \sim L$

$\frac{\delta}{x} \sim \frac{1}{\sqrt{\frac{Ux}{\nu}}} \rightarrow \boxed{\delta \sim \sqrt{x}}$

$$\rightarrow \left(\frac{\nu}{UL}\right)\left(\frac{L}{\delta}\right)^2 \sim 1$$

$$\boxed{\frac{\delta}{L} \sim \frac{1}{\sqrt{Re_L}}}$$

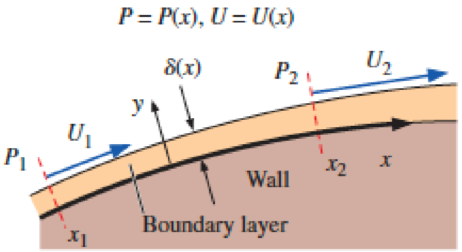
$\rightarrow Re \uparrow \quad \delta \downarrow$



x-momentum boundary layer equation: $$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dP}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

Finally, since we know from the y-momentum equation analysis that the pressure across the boundary layer is the same as that outside the boundary layer, we apply the Bernoulli equation to the outer flow region. Differentiating with respect to x we get

$$\frac{P}{\rho} + \frac{1}{2}U^2 = \text{constant} \quad \rightarrow \quad \frac{1}{\rho}\frac{dP}{dx} = -U\frac{dU}{dx}$$



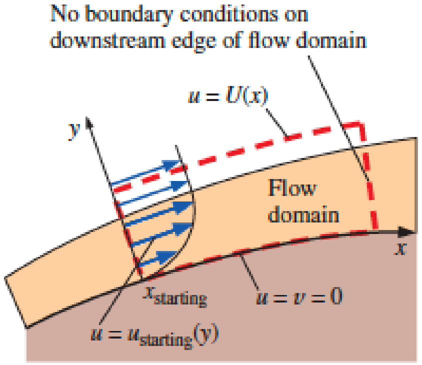
Substitution yields

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

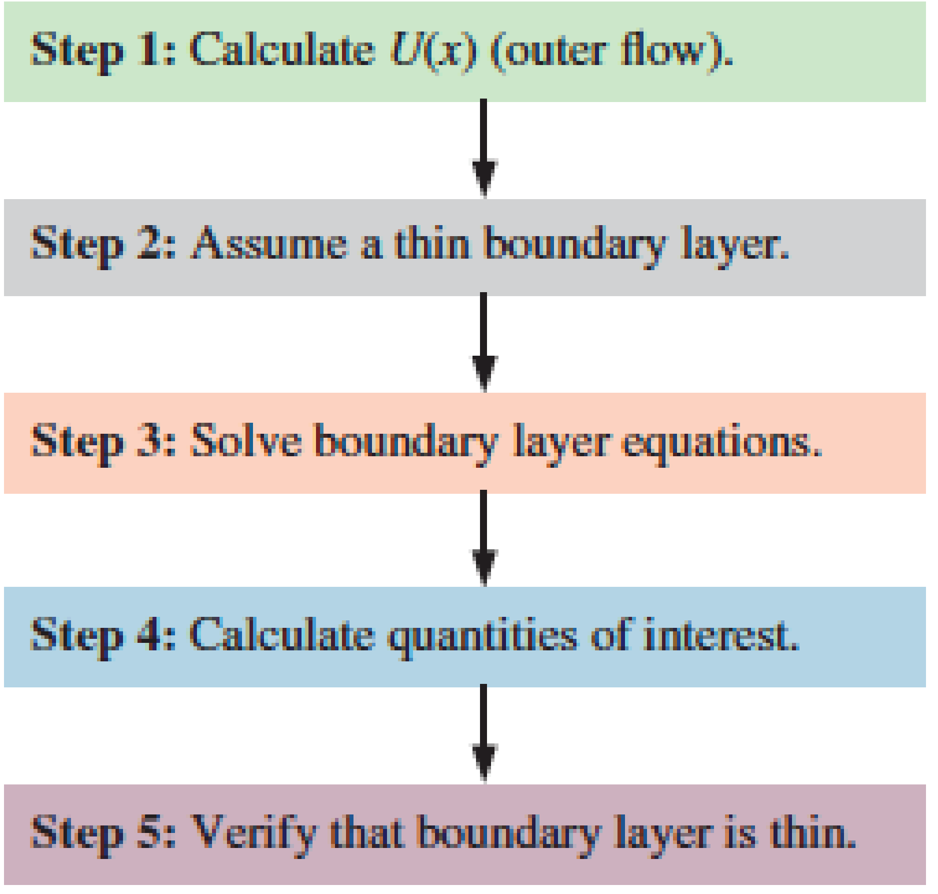
Boundary layer equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

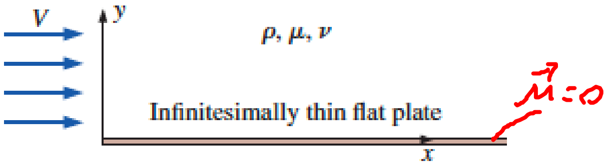


- For a typical boundary layer problem along a wall, we specify the no-slip condition at the wall ($u = v = 0$ at $y = 0$), the outer flow condition at the edge of the boundary layer and beyond [$u = U(x)$ as $y \rightarrow \infty$], and a starting profile at some upstream location [$u = u_{starting}(y)$ at $x = x_{starting}$, where $x_{starting}$ may or may not be zero]. With these boundary conditions, we simply march downstream in the x-direction, solving the boundary layer equations as we go.



Example: Flat plate

$\vec{u}(\infty) = \vec{V}$

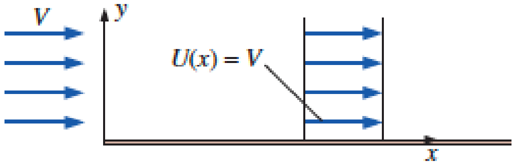


$\vec{u} = 0$

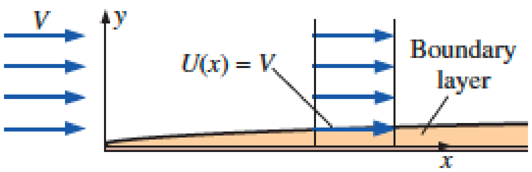
$$\text{Outer flow:} \qquad U(x) = V = \text{constant}$$

$\partial v / \partial x = 0$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$



$$u = 0 \quad \text{at } y = 0 \qquad u = U \quad \text{as } y \to \infty$$
$$v = 0 \quad \text{at } y = 0 \qquad u = U \quad \text{for all } y \text{ at } x = 0$$



No convenient analytical solution is available. However, a series solution was obtained in 1908 by Blasius.

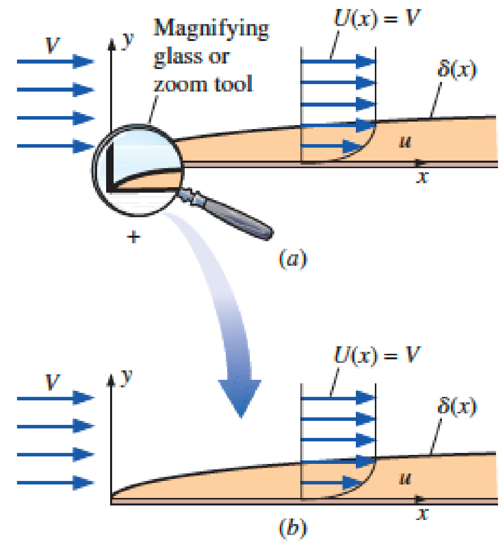
Blasius similarity solution

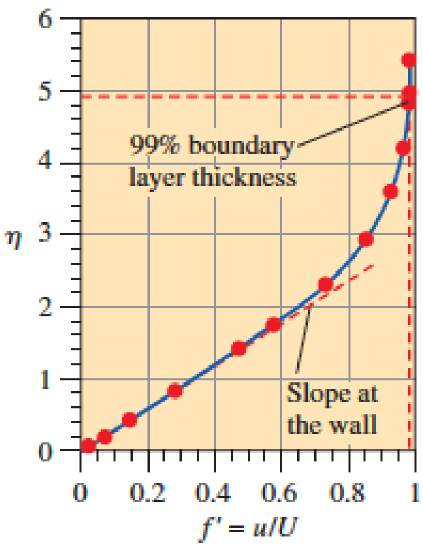
Blasius introduced a **similarity variable** η that combines independent variables x and y into one nondimensional independent variable,

$$\eta = y\sqrt{\frac{U}{\nu x}}$$

and he solved for a nondimensionalized form of the x-component of velocity,

$$f' = \frac{u}{U} = \text{function of } \eta$$





↑ Euler, $\vec{\omega} \approx 0$

↓ N-S

$\vec{\omega} \neq 0$

$$\eta = 4.91 = \sqrt{\frac{U}{\nu x}}\delta \rightarrow \frac{\delta}{x} = \frac{4.91}{\sqrt{\mathrm{Re}_x}}$$

$\Rightarrow \delta \sim \sqrt{x}$

Shear stress in physical variables: $\tau_w = 0.332\frac{\rho U^2}{\sqrt{\mathrm{Re}_x}}$

Blasius solution

Non-linear third order ODE.

Solved numerically or by a series expansion.

Similarity Variable $$\eta = \frac{y}{\delta} = \frac{y}{\sqrt{\nu x / U_0}}$$

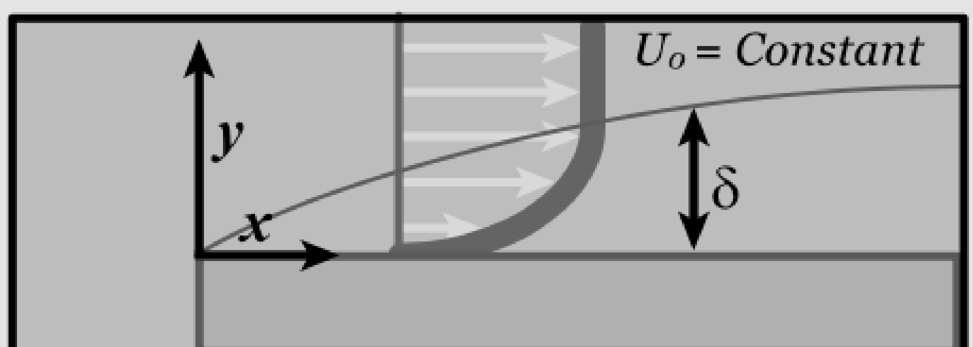
Streamfunction $$f(\eta) = \frac{\psi}{\delta U_0} = \frac{\psi}{\sqrt{\nu x U_0}}$$

Blasius Equation $$f''' + f f'' = 0$$

Boundary Conditions

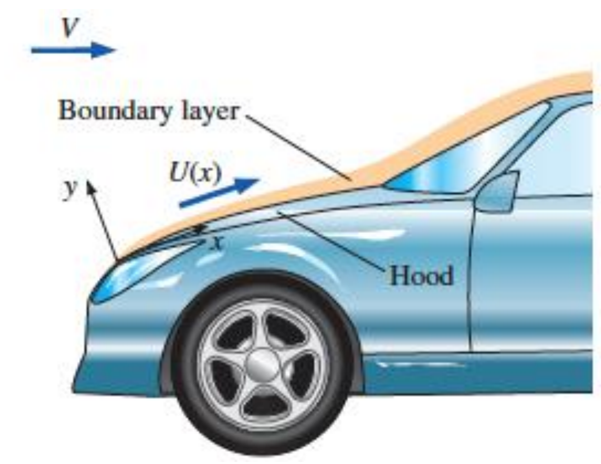
wall: $\eta = 0 \qquad f = f' = 0$

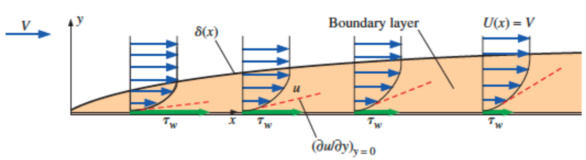
freestream: $\eta \to \infty \qquad f' = 1$

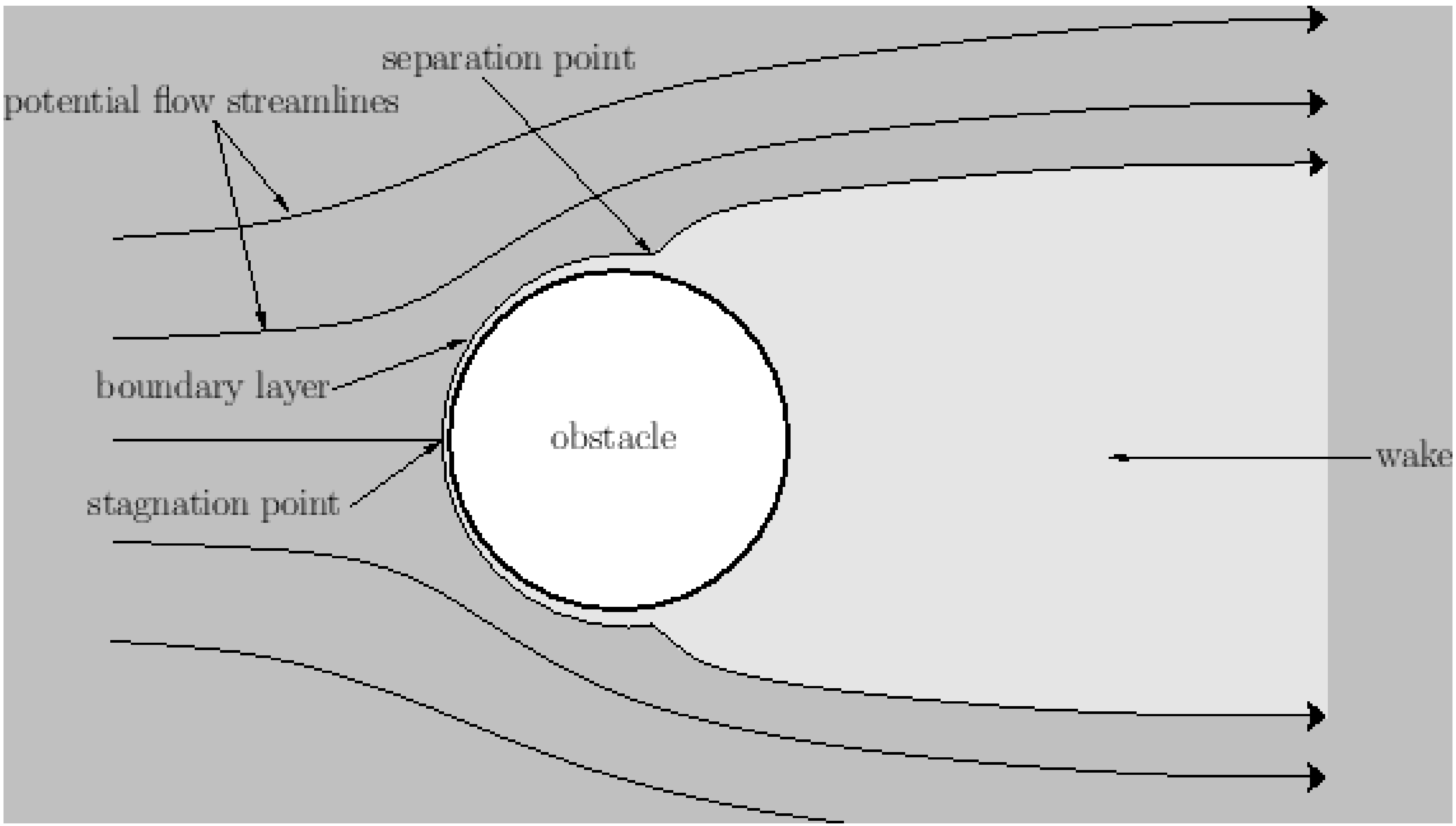


- Discussion: The Blasius boundary layer solution is valid only for flow over a flat plate perfectly aligned with the flow.

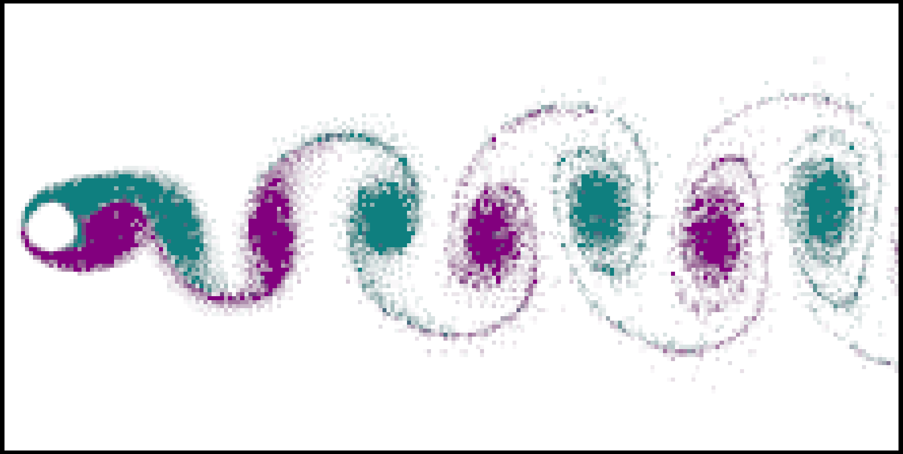
- However, it is often used as a quick approximation for the boundary layer developing along solid walls that are not necessarily flat nor exactly parallel to the flow, as in a car hood.






separation point
potential flow streamlines
boundary layer
obstacle
stagnation point
wake

Backflow, eddies and turbulence



A vortex street around a cylinder. This can occur around cylinders and spheres, for any fluid, cylinder size and fluid speed, provided that the flow has a Reynolds number in the range ~40 to ~1000.

Summary of expressions for laminar and turbulent boundary layers on a smooth flat plate aligned parallel to a uniform stream*

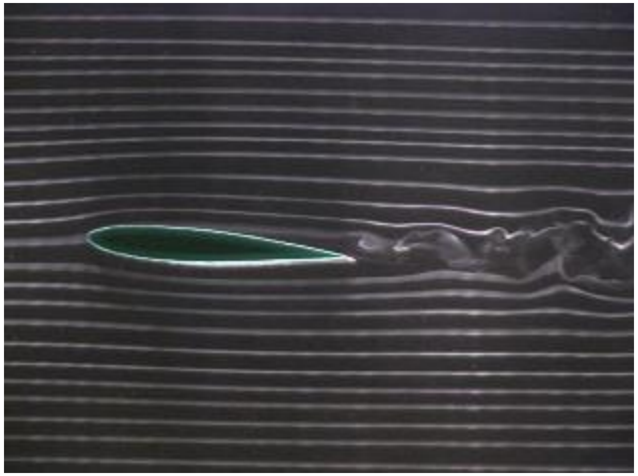
Property	Laminar	(a) Turbulent(†)	(b) Turbulent(‡)
Boundary layer thickness	$\frac{\delta}{x} = \frac{4.91}{\sqrt{\mathrm{Re}_x}}$	$\frac{\delta}{x} \cong \frac{0.16}{(\mathrm{Re}_x)^{1/7}}$	$\frac{\delta}{x} \cong \frac{0.38}{(\mathrm{Re}_x)^{1/5}}$
Displacement thickness	$\frac{\delta^*}{x} = \frac{1.72}{\sqrt{\mathrm{Re}_x}}$	$\frac{\delta^*}{x} \cong \frac{0.020}{(\mathrm{Re}_x)^{1/7}}$	$\frac{\delta^*}{x} \cong \frac{0.048}{(\mathrm{Re}_x)^{1/5}}$
Momentum thickness	$\frac{\theta}{x} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$	$\frac{\theta}{x} \cong \frac{0.016}{(\mathrm{Re}_x)^{1/7}}$	$\frac{\theta}{x} \cong \frac{0.037}{(\mathrm{Re}_x)^{1/5}}$
Local skin friction coefficient	$C_{f,x} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$	$C_{f,x} \cong \frac{0.027}{(\mathrm{Re}_x)^{1/7}}$	$C_{f,x} \cong \frac{0.059}{(\mathrm{Re}_x)^{1/5}}$

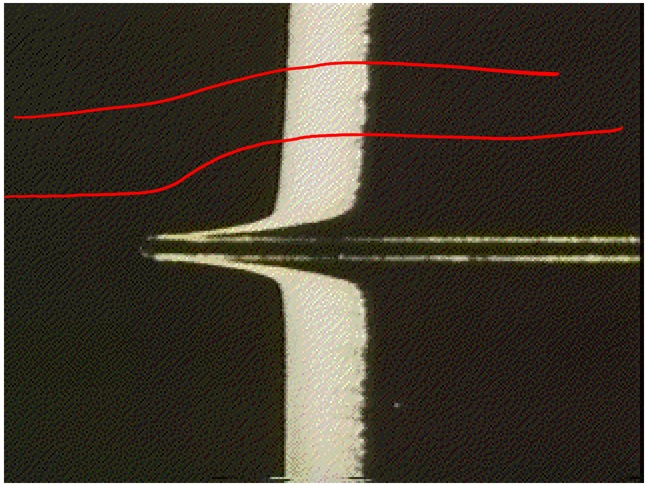
* Laminar values are exact and are listed to three significant digits, but turbulent values are listed to only two significant digits due to the large uncertainty affiliated with all turbulent flow fields.

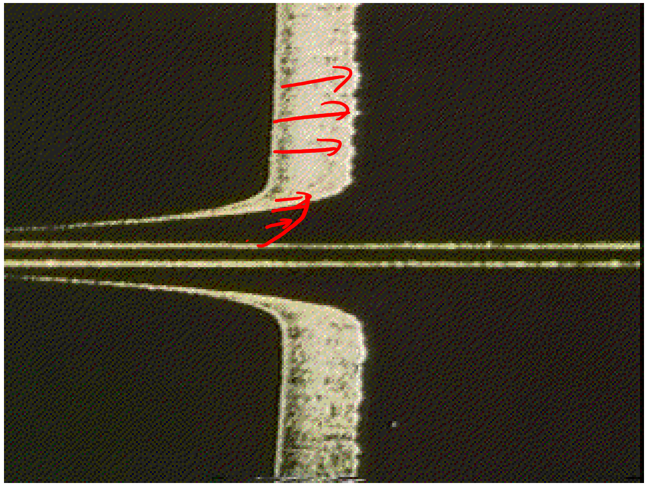
† Obtained from one-seventh-power law.

‡ Obtained from one-seventh-power law combined with empirical data for turbulent flow through smooth pipes.

Local friction coefficient, laminar flat plate: $$C_{f,x} = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$$

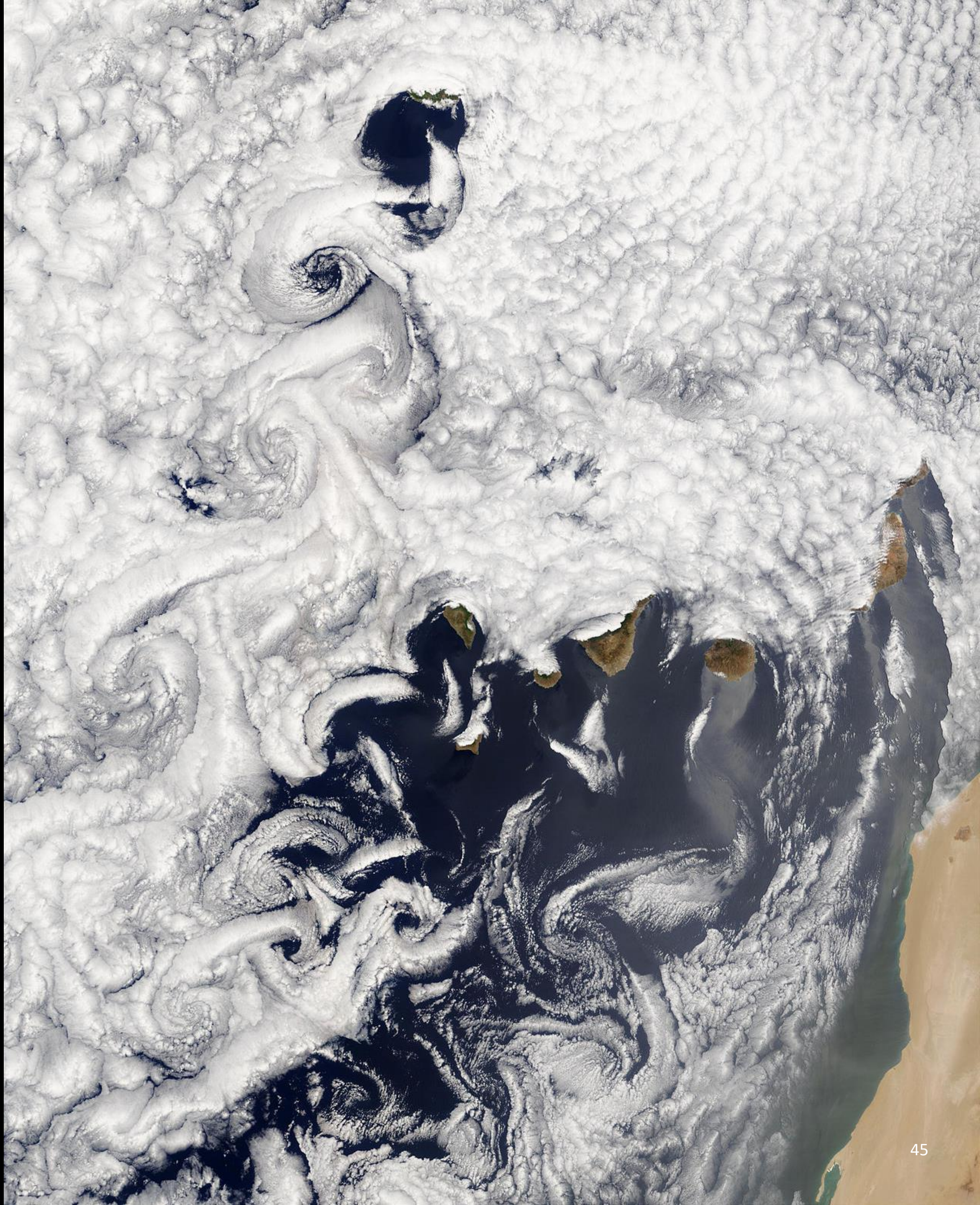






Backflow, eddies and turbulence

Downwind of obstacles, in this case, the Madeira and the Canary Islands off the west African coast, eddies create turbulent patterns called votex streets.



Ex.4, list 6
Acheson, page 269

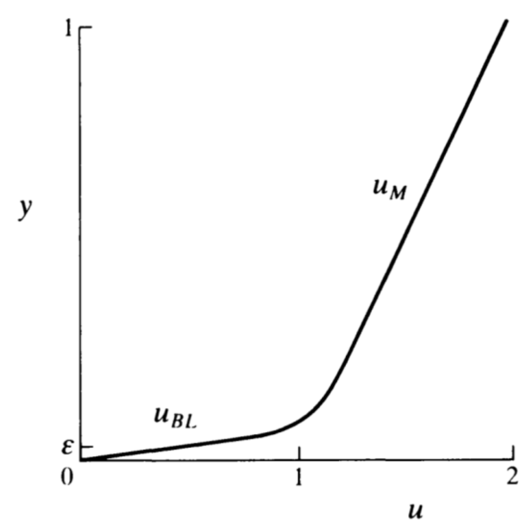
Simplified boundary layer

Consider the following problem for a function $u(y)$:

$$\varepsilon u'' + u' = 1; \qquad u(0) = 0, \quad u(1) = 2, \tag{8.10}$$

where ε denotes a small positive constant. The exact solution is easily shown to be

$$u = y + \frac{1 - e^{-y/\varepsilon}}{1 - e^{-1/\varepsilon}}. \tag{8.11}$$



Now, $e^{-1/\varepsilon}$ is extremely small, and so is $e^{-y/\varepsilon}$, for $0<y<1$, unless y is of order ε. The solution may therefore be approximated, in two parts, by a 'mainstream'

$$u_M = y + 1,$$

and a 'boundary layer' adjacent to $y=0$ with thickness of order ε:

$$u_{BL} = 1 - e^{-y/\varepsilon}.$$

These two expressions represent particular limits of the full, exact solution (8.11), the first being obtained by letting $\varepsilon \to 0$ at fixed y, and the second being obtained by letting $\varepsilon \to 0$ *with y/ε fixed*. Notably,

$$\lim_{y/\varepsilon \to \infty} u_{BL} = \lim_{y \to 0} u_M,$$

and this is the equivalent statement to eqn (8.9) in this elementary example.

It is instructive to take the analogy further by returning to eqn (8.10) and proceeding on an approximate basis from the outset, exploiting the fact that ε is small. If we neglect the term $\varepsilon u''$ entirely, on this basis, we obtain

$$u_0' = 1, \qquad \text{i.e. } u_0 = y + c,$$

and on making this satisfy the condition $u_0(1) = 2$ we obtain an 'outer' solution,

$$u_0(y) = y + 1.$$

This procedure thus far is comparable with treating a high Reynolds number flow as being entirely inviscid; the small parameter ε multiplies the highest derivative in the equation, and by ignoring that term we lower the order of the system and are unable to satisfy all the boundary conditions. Here an 'inner' solution, or boundary layer, is needed near $y = 0$, in order to satisfy the boundary condition there. We may recognize variations of u in this boundary layer to be much more rapid than those elsewhere by changing the independent variable in eqn (8.10) to

$$Y = y/\varepsilon.$$

With this scaling the previously negligible second derivative regains its importance:

$$\varepsilon \frac{1}{\varepsilon^2} \frac{\mathrm{d}^2 u}{\mathrm{d}Y^2} + \frac{1}{\varepsilon} \frac{\mathrm{d}u}{\mathrm{d}Y} = 1,$$

so that to a first approximation the inner solution u_i satisfies

$$\frac{\mathrm{d}^2 u_i}{\mathrm{d}Y^2} + \frac{\mathrm{d}u_i}{\mathrm{d}Y} = 0.$$

This is the equivalent of the boundary layer equation (8.1), in our simple example (and cf. Exercise 8.1). On making the inner solution satisfy the boundary condition $u(0) = 0$ we obtain

$$u_i = A(1 - \mathrm{e}^{-Y}),$$

and the matching condition

$$\lim_{Y \to \infty} u_i = \lim_{y \to 0} u_0$$

determines that $A = 1$. Thus

$$u = \begin{cases} y + 1 & \text{as } \varepsilon \to 0 \text{ for fixed } y, \\ 1 - \mathrm{e}^{-y/\varepsilon} & \text{as } \varepsilon \to 0 \text{ for fixed } y/\varepsilon, \end{cases}$$

in keeping with our deductions from the exact solution (8.11).